Single Pure – Factor & Remainder Theorem

1. The equation $f(x) = x^3 - 4x^2 + x + 6 = 0$ has three integer roots.
   (a) List the values of a for which it is sensible to check whether $f(a) = 0$ and check each of them.
   (b) Solve $f(x) = 0$.
2. By considering f(various sensible values), fully factorise the following:
   (a) $2x^3 + 5x^2 - x - 6$. $(2x+3)(x-1)(x+2)$
   (b) $3x^3 - 2x^2 - 7x - 2$. $(3x+1)(x+1)(x-2)$
   (c) $2x^4 - 9x^3 + x^2 + 12x$. $x(x-4)(2x-3)(x+1)$
   (d) $6x^4 - 7x^3 - 12x^2 + 3x + 2$. $(x-2)(x+1)(2x-1)(3x+1)$
3. Given that $(x - 4)$ is a factor of $2x^3 - 5x^2 - 14x + a$, find a. $a = 8$
4. Given that $(x + 3)$ is a factor of $3x^3 + 9x^2 + bx + 6$, find b. $b = 2$
5. Given that $(2x - 1)$ is a factor of $2x^3 + cx^2 + 8x - 2$, find c. $c = -9$
6. Given that $(x + 3)$ and $(x - 2)$ are factors of $2x^3 + dx^2 + ex - 6$, find d and e. $d = 3, e = -11$
7. Find the remainder when $2x^3 + 4x^2 - x - 1$ is divided by $(x + 1)$. 2
8. Find the remainder when $x^3 - 2x^2 + 3x + 2$ is divided by $(2x - 1)$. $\frac{25}{8}$
9. Given that the remainder when $x^3 + 3x^2 + ax - 3$ is divided by $(x - 1)$ is 3, find a. $a = 2$
10. Given that the remainder when $x^3 + bx^2 + x - 3$ is divided by $(x - 3)$ is -9, find b. $b = -4$
11. Given that the remainder when $cx^3 + 4x^2 + 6x - 3$ is divided by $(2x - 1)$ is 2, find c. $c = 8$
12. Given that $(x - 1)$ is a factor of $f(x) = 2x^3 + \alpha x^2 + \beta x + 2$, and that the remainder when $f(x)$ is divided by $(x - 3)$ is 14, find α and β. $\alpha = -5, \beta = 1$
13. When $x^3 + ax^2 + bx + 8$ is divided by $x - 3$ the remainder is 2 and when it is divided by $x + 1$ the remainder is -2. Find a and b and hence obtain the remainder on dividing by $x - 2$. ☐
14. When $f(x) = 2x^3 + ax^2 + bx + 6$ is divided by $x - 1$ there is no remainder and when $f(x)$ is divided by $x + 1$ the remainder is 10. Find a and b and hence solve the equation $f(x) = 0$.
    $a = -1, b = -7, x = 1$ or $x = \frac{3}{2}$ or $x = -2$
15. The remainder when $x^3 + ax^2 + bx - 1$ is divided by $x - 2$ is 17 more than when it is divided by $x - 1$. The remainder when $x^3 + 2x^2 + 3x - 1$ is divided by $x + 1$ is 7 less than when it is divided by $x - 1$. Find a and b. ☐
16. The polynomial $p(x) = x^3 + ax^2 + bx + c$ leaves remainders $-36, -20, 0$ on division by $x + 1$, $x + 2$, $x + 3$ respectively. Solve the equation $p(x) = 0$. $x = -3$ or $x = -7$ or $x = 2$

17. A cuboidal tank has a square base (of side length x) and maximum volume 8 m^3.

    (a) Write down an expression, in terms of x, for the height of the tank. $\boxed{\frac{8}{x^2}}$

    (b) Show that the surface area of the tank is

    $$\left(x^2 + \frac{32}{x}\right) m^2.$$

    (c) Given that the surface area is $24m^2$ show that $x^3 - 24x + 32 = 0$.

    (d) Solve $x^3 - 24x + 32 = 0$ to find the possible values for x. $\boxed{x = 4 \text{ or } x = 2\sqrt{3} - 2}$

18. Show that $x - y - z$ is a factor of the expression

    $$x^3 + y^3 + z^3 - yz(y + z) - zx(z + x) - xy(x + y) + 2xyz.$$

    Without further working write down two other factors of this expression.

19. Factorise $x^5 - 1$. $\boxed{(x - 1)(x^4 + x^3 + x^2 + x + 1)}$

20. Factorise $x^6 + 1$. $\boxed{(x^2 + 1)(x^4 - x^2 + 1)}$

21. Factorise $x^5 - x^4 - 1$. $\boxed{(x^2 - x + 1)(x^3 - x - 1)}$

22. Factorise $2x^6 - 3x^5 + x^4 - 4x^3 + 4x^2 + x - 1$. $\boxed{(2x^2 - 3x + 1)(x^4 - 2x - 1)}$

23. Factorise $x^7 - 2x^6 + 2x^3 - 1$. $\boxed{(x - 1)(x^3 - x - 1)(x^3 - x^2 - 1)}$

24. STEP. The polynomial $p(x)$ is of degree 9 and $p(x) - 1$ is exactly divisible by $(x - 1)^5$.

    (a) Find the value of $p(1)$.

    (b) Show that $p'(x)$ is exactly divisible by $(x - 1)^4$.

    (c) Given also that $p(x) + 1$ is exactly divisible by $(x + 1)^5$, find $p(x)$.